Fp3 UK June 2016 Model Answers

Kprime 2

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1.

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & -3 \\ k & 1 & 3 \\ 2 & -1 & k \end{pmatrix}, \text{ where } k \text{ is a constant}$$

Given that the matrix **A** is singular, find the possible values of k.

(4)

1.

$$\det A = -2\begin{vmatrix} 1 & 3 \\ -1 & k \end{vmatrix} - \begin{vmatrix} k & 3 \\ 2 & k \end{vmatrix} - 3\begin{vmatrix} k & 1 \\ 2 & -1 \end{vmatrix}$$

$$= -2(k+3) - (k^2-6) - 3(-k-2)$$

$$= -2k - 6 - k^2 + 6 + 3k + 6$$

$$= -k^2 + k + 6 = 0$$

$$\therefore \quad k^2 - k - 6 = 0$$

$$\therefore \quad (k-3)(k+2) = 0$$

$$\therefore \quad \underline{\underline{k = 3}}$$

$$\underline{\underline{k = -2}}$$

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2. The curve C has equation

$$y = \frac{x^2}{8} - \ln x, \quad 2 \leqslant x \leqslant 3$$

Find the length of the curve C giving your answer in the form $p + \ln q$, where p and q are rational numbers to be found.

(7)

2. $y = \dfrac{x^2}{8} - \ln x$

$$\frac{\partial y}{\partial x} = \frac{x}{4} - \frac{1}{x} = \frac{x^2}{4x} - \frac{4}{4x}$$

$$= \frac{x^2-4}{4x}$$

$$\therefore \left(\frac{\partial y}{\partial x}\right)^2 = \frac{(x^2-4)^2}{16x^2}$$

$$\therefore \quad 1 + \left(\frac{\partial y}{\partial x}\right)^2 = 1 + \frac{(x^2-4)^2}{16x^2} = \frac{16x^2 + x^4 - 8x^2 + 16}{16x^2}$$

$$= \frac{x^4 + 8x^2 + 16}{16x^2} = \frac{(x^2+4)^2}{16x^2}$$

$$\therefore \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} = \frac{x^2+4}{\cancel{16x}\,4} \quad \frac{x^2+4}{4x}$$

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$$\therefore S = \int_2^3 \sqrt{1+\left(\frac{\partial x}{\partial x}\right)^2}\ \partial x$$

$$= \int_2^3 \frac{x^2+4}{4x}\ \partial x$$

$$= \int_2^3 \frac{1}{4}x + \frac{1}{x}\ \partial x$$

$$= \left[\frac{1}{8}x^2 + \ln x\right]_2^3 \cancel{\partial x}$$

$$= \frac{9}{8} + \ln 3 - \frac{1}{2} - \ln 2$$

$$= \frac{5}{8} + \ln\frac{3}{2}$$

3. (a) Prove that

$$\frac{\mathrm{d}(\operatorname{arcoth} x)}{\mathrm{d}x} = \frac{1}{1-x^2}$$

(3)

Given that $y = (\operatorname{arcoth} x)^2$,

(b) show that

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{1-x^2}$$

where k is a constant to be determined.

(5)

3(a). Let $y = \operatorname{arcoth} x$

$\cancel{\frac{\partial y}{\partial x} =}$ $\therefore \coth y = x$

$\therefore \frac{\partial y}{\partial x}(-\operatorname{cosech}^2 y) = 1$

$\operatorname{cosech}^2 y \equiv \coth^2 y - 1$

$\therefore \frac{\partial y}{\partial x} \cdot (-(\coth^2 y - 1)) = 1$

$\coth y = x \Rightarrow \coth^2 y = x^2$

$\therefore \frac{\partial y}{\partial x} \times (-(x^2-1)) = 1$

$\therefore \frac{\partial y}{\partial x} = \frac{1}{-(x^2-1)} = \frac{1}{1-x^2}$

as required.

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(b) $y = (\operatorname{arcoth} x)^2$

$$\therefore \frac{\partial y}{\partial x} = \frac{2\operatorname{arcoth} x}{1-x^2} \Rightarrow -2x\frac{\partial y}{\partial x} = \frac{-4x\operatorname{arcoth} x}{1-x^2}$$

$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{(1-x^2)\left(\frac{2}{1-x^2}\right) - 2\operatorname{arcoth} x\,(-2x)}{(1-x^2)^2}$$

$$= \frac{2+4x\operatorname{arcoth} x}{(1-x^2)^2}$$

$$\Rightarrow (1-x^2)\frac{\partial^2 y}{\partial x^2} = (1-x^2)\left(\frac{2+4x\operatorname{arcoth} x}{(1-x^2)^2}\right)$$

$$= \frac{2+4x\operatorname{arcoth} x}{1-x^2}$$

$$\therefore \text{LHS} = (1-x^2)\frac{\partial^2 y}{\partial x^2} - 2x\frac{\partial y}{\partial x} = \frac{2+4x\operatorname{arcoth} x}{1-x^2} - \frac{4x\operatorname{arcoth} x}{1-x^2}$$

$$= \frac{2+4x\operatorname{arcoth} x - 4x\operatorname{arcoth} x}{1-x^2} = \frac{2}{1-x^2}$$

$k = 2$

4. (i) Find, without using a calculator,

$$\int_3^5 \frac{1}{\sqrt{15+2x-x^2}}\,dx$$

giving your answer as a multiple of π.

(5)

(ii)

(a) Show that

$$5\cosh x - 4\sinh x = \frac{e^{2x}+9}{2e^x}$$

(3)

(b) Hence, using the substitution $u = e^x$ or otherwise, find

$$\int \frac{1}{5\cosh x - 4\sinh x}\,dx$$

(4)

4(i). $15+2x-x^2 = -(x^2-2x-15)$

$= -((x-1)^2-16)$

$= 16-(x-1)^2$

$\therefore \int_3^5 \frac{1}{\sqrt{16-(x-1)^2}}\,dx = \left[\arcsin\left(\frac{x-1}{4}\right)\right]_3^5$

$= \arcsin(1) - \arcsin\frac{1}{2}$

$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

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Question 4 continued

(ii) $LHS = 5\cosh x - 4\sinh x = \frac{5e^x + 5e^{-x}}{2} - \frac{4e^x - 4e^{-x}}{2}$

$= \frac{5e^x + 5e^{-x} - 4e^x + 4e^{-x}}{2}$

$= \frac{e^x + 9e^{-x}}{2} = \frac{(e^x + 9e^{-x})e^x}{2e^x}$

$= \frac{e^{2x} + 9}{2e^x} = RHS$ as required.

(b) $u = e^x \Rightarrow \frac{\partial u}{\partial x} = e^x = u \Rightarrow \partial x = \frac{1}{u} \cdot \partial u$

$\therefore \int \frac{1}{5\cosh x - 4\sinh x} \partial x = \int \frac{2e^x}{e^{2x}+9} dx$

$= \int \frac{2u}{u^2 + 9} \cdot \frac{1}{u} \partial u$

$= \int \frac{2}{u^2+9} \partial u = \frac{2}{3} \arctan\left(\frac{u}{3}\right) + C$

$= \frac{2}{3} \arctan\left(\frac{e^x}{3}\right) + C$

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5. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

The point $P\ (4\sec\theta, 3\tan\theta)$, $0 < \theta < \frac{\pi}{2}$, lies on H.

(a) Show that an equation of the normal to H at the point P is

$$3y + 4x\sin\theta = 25\tan\theta$$

(5)

The line l is the directrix of H for which $x > 0$

The normal to H at P crosses the line l at the point Q. Given that $\theta = \frac{\pi}{4}$

(b) find the y coordinate of Q, giving your answer in the form $a + b\sqrt{2}$, where a and b are rational numbers to be found.

(6)

5(a). @ P, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\sec^2\theta}{4\sec\theta\tan\theta}$

$= \frac{3\sec\theta}{4\tan\theta} = \frac{3}{4\sin\theta}$

$\therefore$ gradient of normal $= -\frac{4\sin\theta}{3}$

$y - y_1 = m(x - x_1) \Rightarrow \cancel{y - 4\sec\theta}$

$\cancel{y - 3\tan\theta = \frac{3}{4\sin\theta}(x - 4\sec\theta)}$

$\therefore\ y - 3\tan\theta = -\frac{4\sin\theta}{3}(x - 4\sec\theta)$

$\therefore\ 3y - 9\tan\theta = -4\sin\theta\, x + 16\tan\theta$

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P 4 4 5 6 6 A 0

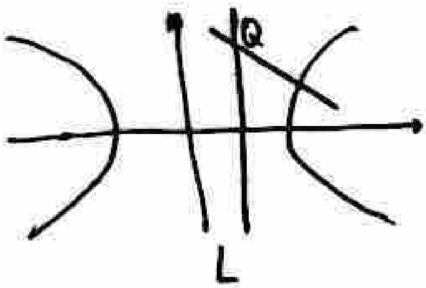
$\therefore 3y + 4x\sin\theta = 25\tan\theta$

as required.

(b) $a^2 = 16$, $b^2 = 9$ $\Big\}$ $9 = 16(e^2 - 1)$

$\therefore e = \frac{5}{4}$

$\theta = \frac{\pi}{4} \Rightarrow 3y + 2\sqrt{2}\,x = 25$



directrix $\frac{a}{e} = \frac{16}{5}$

$x_Q = \frac{16}{5}$

$\Rightarrow 3y + \frac{32}{5}\sqrt{2} = 25$

$\therefore y_Q = \frac{25}{3} - \frac{32}{15}\sqrt{2}$

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6. $$\mathbf{M} = \begin{pmatrix} p & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & q \end{pmatrix}$$

where p and q are constants.

Given that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{M}$,

(a) find the eigenvalue corresponding to this eigenvector, **(3)**

(b) find the value of p and the value of q. **(3)**

Given that 6 is another eigenvalue of $\mathbf{M}$,

(c) find a corresponding eigenvector. **(2)**

Given that $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is a third eigenvector of $\mathbf{M}$ with eigenvalue 3

(d) find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}$$

(3)

6(a). $\begin{pmatrix} p & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & q \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \lambda\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 2p+4 \\ -18 \\ 4+q \end{pmatrix} = \begin{pmatrix} 2\lambda \\ -2\lambda \\ \lambda \end{pmatrix} \Rightarrow -18 = -2\lambda$

$\therefore \quad \lambda = 9$

Question 6 continued

(b) $2p+4=2\lambda \Rightarrow 2p+4=18$

$\therefore p=7$

$4+q=9 \Rightarrow \quad q=5$

(c) $\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\therefore \begin{pmatrix} 7x-2y \\ -2x+6y-2z \\ -2y+5z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$

$\left.\begin{array}{l} 7x-2y=6x \Rightarrow x=2y \\ -2y+5z=6z \Rightarrow z=-2y \end{array}\right\} \Rightarrow \underline{x=2y=-z}$

Let $x=2 \Rightarrow y=1 \Rightarrow z=-2$

$\therefore$ E-Vector is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

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Question 6 continued

(d) $\lambda = 9$ & $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Normalise $\Rightarrow \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$\lambda = 6$ & $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Normalise $\Rightarrow \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

$\lambda = 3$ & $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ Normalise $\Rightarrow \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\therefore P = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$

$D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

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7. Given that

$$I_n = \int \frac{\sin nx}{\sin x}\,dx, \quad n \geqslant 1$$

(a) prove that, for $n \geqslant 3$

$$I_n - I_{n-2} = \int 2\cos(n-1)x\,dx$$

(3)

(b) Hence, showing each step of your working, find the exact value of

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\sin 5x}{\sin x}\,dx$$

giving your answer in the form $\frac{1}{12}(a\pi + b\sqrt{3} + c)$, where a, b and c are integers to be found.

(7)

$$I_n - I_{n-2} = \int \frac{\sin nx}{\sin x}\,dx - \int \frac{\sin[(n-2)x]}{\sin x}\,dx$$

$$= \int \frac{\sin(nx) - \sin(nx-2x)}{\sin x}\,dx$$

$$= \int \frac{2\cos\left(\frac{2nx-2x}{2}\right)\sin\left(\frac{2x}{2}\right)}{\sin x}\,dx \quad *$$

$$= \int 2\cos(n-1)x\,dx$$

$*$ Using $\sin A - \sin B \equiv 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

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Question 7 continued

$$I_5 - I_3 = 2\int_{\pi/12}^{\pi/6} \cos 4x \, dx$$

$$= 2\left[\tfrac{1}{4}\sin 4x\right]_{\pi/12}^{\pi/6}$$

$$= 0 \Rightarrow I_5 = I_3$$

$$I_3 - I_1 = \int_{\pi/12}^{\pi/6} 2\cos 2x \, dx$$

$$= \left[\sin 2x\right]_{\pi/12}^{\pi/6}$$

$$= \frac{-1+\sqrt{3}}{2}$$

$$I_1 = \int_{\pi/12}^{\pi/6} 1 \, dx = \frac{\pi}{6} - \frac{\pi}{12} = \frac{\pi}{12}$$

$$\therefore I_3 = \frac{\pi}{12} + \frac{-1+\sqrt{3}}{2} = \frac{\pi}{12} + \frac{-6+6\sqrt{3}}{12}$$

$$\Rightarrow I_5 = \frac{1}{12}\left(\pi + 6\sqrt{3} - 6\right)$$

$a = 1$
$b = 6$
$c = -6$

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8. The plane Π_1 has equation

$$x - 5y - 2z = 3$$

The plane Π_2 has equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$$

where λ and μ are scalar parameters.

(a) Show that Π_1 is perpendicular to Π_2 **(4)**

(b) Find a cartesian equation for Π_2 **(2)**

(c) Find an equation for the line of intersection of Π_1 and Π_2 giving your answer in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors to be found. **(6)**

8(a) Π_2:

$$\underset{\sim}{n}_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{matrix} 1 & 4 & 3 & & 1 & 4 & 3 \\ 2 & -1 & 1 & & 2 & -1 & 1 \end{matrix}$$

$$= \begin{pmatrix} 7 \\ 5 \\ -9 \end{pmatrix}$$

If Π_1 & Π_2 are perpendicular then their normals must be perpendicular.

Π_1: $\underset{\sim}{n}_1 = \begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix}$

$$\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2 = \begin{pmatrix} 7 \\ 5 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix} = 7 - 25 + 18 = 0$$

$\Rightarrow$ Π_1 & Π_2 are perpendicular.

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Question 8 continued

(b) $\underset{\sim}{r}\cdot\begin{pmatrix}7\\5\\-9\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}7\\5\\-9\end{pmatrix}$

$$= 8$$

$$\Rightarrow \underline{\underline{7x+5y-9z=8}}$$

(c) $x-5y-2z=3 \Rightarrow 5y = x-2z-3$

$$7x+5y-9z=8$$

$$\therefore 7x+x-2z-3-9z=8$$

$$\therefore 8x-11z=11$$

$$\therefore \underline{x = \frac{11}{8}+\frac{11}{8}z}$$

$$\therefore 5y = \frac{11}{8}+\frac{11}{8}z-2z-3$$

$$5y = -\frac{13}{8}-\frac{5}{8}z$$

$$\therefore \underline{y = -\frac{13}{40}-\frac{1}{8}z}$$

$z = z$

$$\therefore \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}\frac{11}{8}\\-\frac{13}{40}\\0\end{pmatrix} + z\begin{pmatrix}11/8\\-1/8\\1\end{pmatrix}$$

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Question 8 continued

$$= \begin{pmatrix} 11/8 \\ -13/40 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 11 \\ -1 \\ 8 \end{pmatrix}$$

$$\therefore \left(\underset{\sim}{r} - \begin{pmatrix} \frac{11}{8} \\ -13/40 \\ 0 \end{pmatrix} \right) \times \begin{pmatrix} 11 \\ -1 \\ 8 \end{pmatrix} = 0$$

Note: There are many possible solutions for this part of the question.

Solutions will vary depending on what you expressed your x, y, z in terms of ...